

Effective Hamiltonians for Complexes of Unstable Particles

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Abstract. Effective Hamiltonians governing the time evolution in a subspace of unstable states can be found using more or less accurate approximations. A convenient tool for deriving them is the evolution equation for a subspace of state space sometime called the Krokowski-Rzewuski (KR) equation. KR equation results from the Schrödinger equation for the total system under considerations. We will discuss properties of approximate effective Hamiltonians derived using KR equation for n -particle, two particle and for one particle subspaces. In a general case these effective Hamiltonians depend on time t . We show that at times much longer than times at which the exponential decay take place the real part of the exact effective Hamiltonian for the one particle subsystem (that is the instantaneous energy) tends to the minimal energy of the total system when $t \rightarrow \infty$ whereas the imaginary part of this effective Hamiltonian tends to the zero as $t \rightarrow \infty$.

1. Introduction

The standard approach to searching for the properties of subsystems of unstable particles makes use of more or less accurate approximate methods to solve evolution equation for such subsystems. A typical example of such methods are Weisskopf–Wigner (WW) approximation [1] or Lee–Oehme–Yang (LOY) approximation [2, 3]. All intermediate steps of WW or LOY approximations leading to the final formulae describing the time evolution of unstable particles are rather far from mathematical precision. What is more, attempts to confront the predicted properties of the considered systems, obtained within the use of such approximate methods, with those following from the analytical properties of the exact solutions of the quantum evolution equation are rather sporadic. These analytical properties can be extracted from properties of the transition amplitudes

$$A_{\alpha\beta}(t) = \langle \alpha | U(t) | \beta \rangle, \quad (1)$$

where $|\alpha\rangle, |\beta\rangle \in \mathcal{H}$, \mathcal{H} is the Hilbert state space of the total system considered, and $U(t)$ is the total unitary evolution equation solving the Schrödinger

equation

$$i\frac{\partial}{\partial t}U(t)|\psi\rangle = HU(t)|\psi\rangle, \quad U(0) = I, \quad (2)$$

(we use $\hbar = c = 1$ units), I is the unit operator in \mathcal{H} , $|\psi\rangle \equiv |\psi; t_0 = 0\rangle \in \mathcal{H}$ is the initial state of the system, (in our case $|\psi; t\rangle = U(t)|\psi\rangle$), and H is the total (selfadjoint) Hamiltonian, acting in \mathcal{H} .

Amplitudes $A_{\alpha\beta}(t)$ can be expressed in terms of the energy (mass) densities $\omega_{\alpha\beta}(E)$ as follows

$$A_{\alpha\beta}(t) = \int_{\text{Spec}(H)} \omega_{\alpha\beta}(E) e^{-iEt} dE, \quad (3)$$

Assuming that the exact properties of real systems containing unstable particles are described by the exact solutions of Eq. (2), properties of amplitudes $A_{\alpha\beta}(t)$ following, eg., from symmetries of the system under considerations can be used to examine properties of some parameters describing the unstable particles and obtained by means of $A_{\alpha\beta}(t)$ following from approximate methods of calculations. A typical example is a subsystem of neutral mesons and properties of $A_{\alpha\beta}(t)$ following from \mathcal{CP} or \mathcal{CPT} invariance of H , [4]. Moreover amplitudes $A_{\alpha\beta}(t)$ are convenient for numerical simulations of time evolution of the states considered: It is sufficient to assume the form of the densities $\omega_{\alpha\beta}(E)$ and then to use computer methods to find $A_{\alpha\beta}(t)$ as a function of time t .

In the case of neutral kaons (neutral mesons in general) all known properties including CP- and hypothetically possible CPT-violation effects in such complexes are described by solving the Schrödinger-like evolution equation [2] — [6],

$$i\frac{\partial}{\partial t}|\psi; t\rangle_{\parallel} = H_{\parallel}|\psi; t\rangle_{\parallel} \quad (4)$$

for $|\psi; t\rangle_{\parallel}$ belonging to the subspace $\mathcal{H}_{\parallel} \subset \mathcal{H}$, e.g., spanned by orthonormal neutral kaons states $|K_0\rangle = |\mathbf{1}\rangle$, $|\bar{K}_0\rangle = |\mathbf{2}\rangle \in \mathcal{H}$, $\langle \mathbf{j} | \mathbf{k} \rangle = \delta_{jk}$, $j, k = 1, 2$, (then states corresponding to the decay products belong to $\mathcal{H} \ominus \mathcal{H}_{\parallel} \stackrel{\text{def}}{=} \mathcal{H}_{\perp}$), and nonhermitian effective Hamiltonian H_{\parallel} obtained usually by means of the LOY approach (within the WW approximation) [2, 3, 6],

$$H_{\parallel} \equiv M - \frac{i}{2}\Gamma, \quad (5)$$

where H_{\parallel} , $M = M^+$, $\Gamma = \Gamma^+$ are (2×2) matrices, M is the mass matrix and Γ is the decay matrix.

Within the WW approximation for a single particle subsystem the evolution equation has a similar form to (4),

$$i\frac{\partial a(t)}{\partial t} = h_{WW} a(t), \quad (6)$$

where $|\psi; t\rangle_{\parallel} \equiv a(t) |\alpha\rangle$ and $\dim \mathcal{H}_{\parallel} = 1$, $a(0) = 1$, and $h_{WW} = E_{\alpha}^0 - \frac{i}{2} \gamma_{\alpha}^0$ is the WW effective hamiltonian governing the time evolution in one dimensional subspace of states (E_{α}^0 is the energy of the system in the state $|\alpha\rangle$ and γ_{α}^0 is the decay width). In this case the amplitude $A_{\alpha\beta}(t)$, (3), can be replaced by $a(t) \stackrel{\text{def}}{=} A_{\alpha\alpha}(t)$ and $\omega_{\alpha\beta}(E)$ by $\omega(E) \stackrel{\text{def}}{=} \omega_{\alpha\alpha}(E)$, and here $\omega(E) \geq 0$ [7].

The analysis of the models of the decay processes shows that the LOY and WW effective Hamiltonians appearing in (4) and (6) describe properties of two particle or single particle complexes to a very high accuracy for a wide time range t : From t suitably later than some $T_0 \simeq t_0 = 0$ but $T_0 > t_0$ up to $t \gg \tau = 1/\gamma_{\alpha}^0$ and smaller than the transition time $t = t_{as}$, where t_{as} denotes the time t for which the nonexponential deviations of the survival probability begin to dominate.

In [8] assuming that the spectrum of H must be bounded from below, ($\text{Spec.}(H) = [E_{min}, +\infty) > -\infty$), and using the Paley–Wiener Theorem [9] it was proved that in the case of unstable states there must be

$$|a(t)| \geq B e^{-b t^q}, \quad (7)$$

for $|t| \rightarrow \infty$. Here $B > 0$, $b > 0$ and $0 < q < 1$. This means that the decay law $\mathcal{P}(t) = |a(t)|^2$ of unstable states decaying in the vacuum can not be described by an exponential function of time t if time t is suitably long, $t \rightarrow \infty$, and that for these lengths of time $\mathcal{P}(t)$ tends to zero as $t \rightarrow \infty$ more slowly than any exponential function of t [10]. From the model analysis it follows that in the general case the decay law $\mathcal{P}(t)$ takes the inverse power-like form $t^{-\lambda}$, (where $\lambda > 0$), for suitably large $t \geq t_{as} \gg \tau$. Not long ago this effect was confirmed experimentally: in the experiment described in [11], the evidence of deviations from the exponential decay law at long times was reported. The conclusion is that the LOY, WW and similar effective Hamiltonians can not be used when one analyzes a very long time properties of unstable systems.

The aim of this paper is to analyze more accurate approximations for the effective Hamiltonians governing the time evolution in subspace of unstable states than those given LOY or WW formulae and to analyze very long time properties of the effective hamiltonian for an one particle subsystem.

2. Beyond the WW and LOY approximations

2.1. APPROXIMATE FORMULAE FOR H_{\parallel} — A GENERAL CASE

The approximate formulae for $H_{\parallel} \equiv H_{\parallel}(t)$ have been derived in [12] — [16] using the Krolkowski–Rzewuski equation for the projection of a state vector [17], which results from the Schrödinger equation (2) for the total

system under consideration, and, in the case of the initial conditions of the type $|\psi\rangle \equiv |\psi\rangle_{\parallel}$, $|\psi\rangle_{\perp} = |\psi\rangle - |\psi\rangle_{\parallel} = 0$, takes the following form

$$(i\frac{\partial}{\partial t} - PHP)U_{\parallel}(t)|\psi\rangle_{\parallel} = -i \int_0^{\infty} K(t-\tau)U_{\parallel}(\tau)|\psi\rangle_{\parallel} d\tau, \quad (8)$$

where $U_{\parallel}(0) = P$, $K(t) = \Theta(t)PHQ \exp[-itQHQ]QHP$, and $\Theta(t) = \{1 \text{ for } t \geq 0, 0 \text{ for } t < 0\}$ is the unit step function.

The integro-differential equation (8) can be replaced by the following differential one (see [12] — [17])

$$(i\frac{\partial}{\partial t} - PHP - V_{\parallel}(t))U_{\parallel}(t)|\psi\rangle_{\parallel} = 0, \quad (9)$$

where

$$PHP + V_{\parallel}(t) \stackrel{\text{def}}{=} H_{\parallel}(t). \quad (10)$$

Taking into account (8) and (9) or (4) one finds from (8)

$$V_{\parallel}(t)U_{\parallel}(t) = -i \int_0^{\infty} K(t-\tau)U_{\parallel}(\tau) d\tau \stackrel{\text{def}}{=} -iK * U_{\parallel}(t). \quad (11)$$

(Here the asterisk, $*$, denotes the convolution: $f * g(t) = \int_0^{\infty} f(t-\tau)g(\tau) d\tau$). Next, using this relation and a retarded Green's operator $G(t)$ for the equation (8)

$$G(t) = -i\Theta(t) \exp(-itPHP)P, \quad (12)$$

one obtains [15, 16]

$$U_{\parallel}(t) = \left[1_* + \sum_{n=1}^{\infty} (-i)^n L * \dots * L \right] * U_{\parallel}^{(0)}(t), \quad (13)$$

where L is convoluted n times, $1_* \equiv 1_*(t) \equiv \delta(t)$, $L(t) = G * K(t)$, and

$$U_{\parallel}^{(0)} = \exp(-itPHP) P \quad (14)$$

is a "free" solution of Eq. (8). Thus from (11)

$$V_{\parallel}(t) U_{\parallel}(t) = -iK * \left[1_* + \sum_{n=1}^{\infty} (-i)^n L * \dots * L \right] * U_{\parallel}^{(0)}(t), \quad (15)$$

Of course, the series (13), (15) are convergent if $\|L(t)\| < 1$. If for every $t \geq 0$

$$\|L(t)\| \ll 1, \quad (16)$$

then, to the lowest order of $L(t)$, one finds from (15) [15, 16]

$$V_{\parallel}(t) \cong V_{\parallel}^{(1)}(t) \stackrel{\text{def}}{=} -i \int_0^\infty K(t-\tau) \exp[i(t-\tau)PHP] d\tau. \quad (17)$$

Note that from the definition of $L(t)$ it follows that

$$L(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow 0, \quad (18)$$

which means that the condition (16) is always fulfilled for $t \rightarrow 0$ and thus $V_{\parallel}(t)$ given by formula (17) describes very well properties of the subsystem under considerations for $t \rightarrow 0$.

2.2. n -DIMENSIONAL CASE

Now let us consider a general case of n -dimensional subspace \mathcal{H}_{\parallel} . Vectors from such subspaces describe states of n -level (n -particle) subsystems. The only problem is to calculate $P \exp[itPHP]$ in (17) for the case of $\dim(\mathcal{H}_{\parallel}) = n$. Note that it is convenient to consider such \mathcal{H}_{\parallel} as the subspace spanned by a set of orthonormal vectors $\{|\mathbf{e}_j\rangle\}_{j=1}^n \in \mathcal{H}$, $\langle \mathbf{e}_j | \mathbf{e}_k \rangle = \delta_{jk}$. Then the projection operator P defining this subspace can be expressed as follows

$$P = \sum_{j=1}^n |\mathbf{e}_j\rangle \langle \mathbf{e}_j|. \quad (19)$$

The operator PHP is selfadjoint, so the $(n \times n)$ matrix representing PHP in the subspace \mathcal{H}_{\parallel} is Hermitian matrix. Solving the eigenvalue problem for this matrix,

$$PHP|\lambda_j\rangle = \lambda_j|\lambda_j\rangle, \quad (j=1,2,\dots,n), \quad (20)$$

one obtains the eigenvalues $\lambda_j = \lambda_j^*$, and eigenvectors $|\lambda_j\rangle$, ($j = 1, 2, \dots, n$). Here for simplicity we assume that $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n \neq \lambda_1 \neq \dots$, etc.. In other words it is assumed that all λ_j are nondegenerate and thus all $|\lambda_j\rangle$ must be orthogonal,

$$\langle \lambda_j | \lambda_k \rangle = \langle \lambda_j | \lambda_j \rangle \delta_{jk}, \quad (j,k=1,2,\dots,n). \quad (21)$$

By means of these eigenvectors one can define new projection operators,

$$P_j \stackrel{\text{def}}{=} \frac{1}{\langle \lambda_j | \lambda_j \rangle} |\lambda_j\rangle \langle \lambda_j|, \quad (j=1,2,\dots,n). \quad (22)$$

The property (21) of the solution of the eigenvalue problem for PHP considered implies that

$$P_j P_k = P_j \delta_{jk}, \quad (j=1,2,\dots,n), \quad (23)$$

and that the completeness requirement for the subspace \mathcal{H}_\parallel

$$\sum_{j=1}^n P_j = P, \quad (24)$$

holds. Now, using the projectors P_j one can write

$$PHP = \sum_{j=1}^n \lambda_j P_j, \quad (25)$$

and

$$Pe^{+itPHP} = P \sum_{j=1}^n e^{+it\lambda_j} P_j. \quad (26)$$

This last relation is the solution for the problem of finding $P \exp[itPHP]$ in the considered case of nondegenerate λ_j and together with the formula (17) for $V_\parallel(t)$ yields

$$V_\parallel^{(1)}(t) = - \sum_{j=1}^n PHQ \frac{e^{-it(QHQ - \lambda_j)} - 1}{QHQ - \lambda_j} QHP P_j, \quad (27)$$

which leads to $V_\parallel \stackrel{\text{def}}{=} \lim_{t \rightarrow \infty} V_\parallel^{(1)}(t)$,

$$V_\parallel = - \sum_{j=1}^n \Sigma(\lambda_j) P_j, \quad (28)$$

where

$$\Sigma(\epsilon) = PHQ \frac{1}{QHQ - \epsilon - i0} QHP. \quad (29)$$

This solves the problem of finding the effective Hamiltonian

$$H_\parallel \equiv PHP + V_\parallel, \quad (30)$$

(where $V_\parallel = \lim_{t \rightarrow \infty} V_\parallel^{(1)}(t)$) governing the time evolution in the n -state subspace \mathcal{H}_\parallel of the total state space \mathcal{H} .

The simplest case is when the operator PHP has n -fold degenerate eigenvalue λ_0 , that is when $\lambda_1 = \lambda_2 = \dots = \lambda_n \stackrel{\text{def}}{=} \lambda_0$. Then

$$V_\parallel^{(1)}(t) = -PHQ \frac{e^{-it(QHQ - \lambda_0)} - 1}{QHQ - \lambda_0} QHP, \quad (31)$$

which gives

$$V_{||} = -\Sigma(\lambda_0). \quad (32)$$

The most interesting cases seem to be the cases when the eigenvalues λ_j of PHP are k -fold degenerate, where $k < n$. Then the form of $V_{||}$ differs from (28) and (32).

So, let $\lambda_1, \lambda_2, \dots, \lambda_k$ be the nondegenerate eigenvalues for PHP and $\lambda_{k+1} = \lambda_{k+2} = \dots = \lambda_n \stackrel{\text{def}}{=} \lambda$. Then

$$PHP = \sum_{j=1}^k \lambda_j P_j + \lambda(P - \sum_{j=1}^k P_j), \quad (33)$$

(here P_j is given by the formula (22)) and

$$Pe^{+itPHP} = P \sum_{j=1}^k e^{+it\lambda_j} P_j + P(P - \sum_{j=1}^k P_j) e^{it\lambda}. \quad (34)$$

Using this last relation and the general formula (17) for $V_{||}(t)$ and then taking $t \rightarrow \infty$ one finds

$$V_{||} = - \sum_{j=1}^k \Sigma(\lambda_j) P_j - \Sigma(\lambda)(P - \sum_{j=1}^k P_j). \quad (35)$$

2.3. 2-DIMENSIONAL CASE

Let us pass on to $n = 2$ case, i.e. to the case of two-dimensional subspace $\mathcal{H}_{||}$, which can be applied to neutral meson complexes. So, if $|\mathbf{e}_j\rangle = |\mathbf{j}\rangle$, ($j = 1, 2$), then the projector P is defined by

$$P \equiv |\mathbf{1}\rangle\langle\mathbf{1}| + |\mathbf{2}\rangle\langle\mathbf{2}|. \quad (36)$$

In the LOY approach it is assumed that vectors $|\mathbf{1}\rangle$, $|\mathbf{2}\rangle$ considered above are eigenstates of $H^{(0)}$ for a 2-fold degenerate eigenvalue m_0 :

$$H^{(0)}|\mathbf{j}\rangle = m_0|\mathbf{j}\rangle, \quad j = 1, 2, \quad (37)$$

where $H^{(0)}$ is the so called free Hamiltonian, $H^{(0)} \equiv H_{strong} = H - H_W$, and H_W denotes weak and other interactions which are responsible for transitions between the eigenvectors of $H^{(0)}$, i.e., for the decay process.

If H has the following property

$$PHP \equiv m_0 P, \quad (38)$$

that is for $H_{12} = H_{21} = 0$, the approximate formula (17) for $V_{\parallel}(t)$ leads to the following form of Pe^{itPHP} ,

$$Pe^{itPHP} = Pe^{itm_0}, \quad (39)$$

and thus to

$$V_{\parallel}^{(1)}(t) = -PHQ \frac{e^{-it(QHQ - m_0)} - 1}{QHQ - m_0} QHP, \quad (40)$$

which leads to

$$V_{\parallel} = \lim_{t \rightarrow \infty} V_{\parallel}^{(1)}(t) = -\Sigma(m_0). \quad (41)$$

This means that in the case (38)

$$H_{\parallel} = m_0 P - \Sigma(m_0), \quad (42)$$

and $H_{\parallel} = H_{LOY}$.

On the other hand, in the case

$$H_{12} = H_{21}^* \neq 0, \quad (43)$$

the form of Pe^{itPHP} is much more complicated.

In the general case (43) one finds the following expressions for the matrix elements $v_{jk}(t \rightarrow \infty) \stackrel{\text{def}}{=} v_{jk}$ of V_{\parallel} [15, 16],

$$\begin{aligned} v_{j1} = & - \frac{1}{2} \left(1 + \frac{H_z}{\kappa} \right) \Sigma_{j1}(H_0 + \kappa) - \frac{1}{2} \left(1 - \frac{H_z}{\kappa} \right) \Sigma_{j1}(H_0 - \kappa) \\ & - \frac{H_{21}}{2\kappa} \Sigma_{j2}(H_0 + \kappa) + \frac{H_{21}}{2\kappa} \Sigma_{j2}(H_0 - \kappa), \\ v_{j2} = & - \frac{1}{2} \left(1 - \frac{H_z}{\kappa} \right) \Sigma_{j2}(H_0 + \kappa) - \frac{1}{2} \left(1 + \frac{H_z}{\kappa} \right) \Sigma_{j2}(H_0 - \kappa) \\ & - \frac{H_{12}}{2\kappa} \Sigma_{j1}(H_0 + \kappa) + \frac{H_{12}}{2\kappa} \Sigma_{j1}(H_0 - \kappa), \end{aligned} \quad (44)$$

where $j, k = 1, 2$, $H_z = \frac{1}{2}(H_{11} - H_{22})$, $H_0 \stackrel{\text{def}}{=} \frac{1}{2}(H_{11} + H_{22})$ and $\kappa = (|H_{12}|^2 + H_z^2)^{1/2}$. Hence, by (10), $h_{jk} = H_{jk} + v_{jk}$. It should be emphasized that all components of the expressions (44) are of the same order with respect to $\Sigma(\varepsilon)$.

Formulae (44) for matrix elements v_{jk} become much simpler if $|H_{12}| \ll |H_0|$. Also the symmetries of the system lead to simpler form of v_{jk} (see eg. system of neutral mesons in which CPT-symmetry is assumed to hold [18]).

2.4. 1-DIMENSIONAL CASE

This is the simplest case. Here $|\mathbf{e}_1\rangle \equiv |\alpha\rangle$, and $P_1 = |\alpha\rangle\langle\alpha|$. This leads to

$$H_{||}(t) = h_{WW}(t) P_1, \quad P_1 H P_1 = E_1 P_1, \quad V_{||}(t) = v_{WW}(t) P_1, \quad (45)$$

where $E_1 = \langle\alpha|H|\alpha\rangle$ and

$$v_{WW}(t) = v_{Ww}^{(1)}(t) = -\langle\alpha|H Q \frac{e^{-it(QHQ - E_1)} - 1}{QHQ - E_1} QH|\alpha\rangle. \quad (46)$$

Thus

$$v_{WW} = \lim_{t \rightarrow \infty} v_{Ww}^{(1)}(t) = -\Sigma_1(E_1), \quad (47)$$

which gives the WW effective Hamiltonian h_{ww} appearing in (6).

3. One-particle effective Hamiltonian at long time region

From (6) one can conclude that the exact one-dimensional effective Hamiltonian, $h(t)$, fulfils the following identity (see [14, 19])

$$h(t) \equiv i \frac{1}{a(t)} \frac{\partial a(t)}{\partial t}, \quad (48)$$

where

$$a(t) \equiv \langle\alpha|e^{-itH}|\alpha\rangle \equiv \int_{Spec.(H)} \omega(E) e^{-iEt} dE, \quad (49)$$

is the survival amplitude.

In general, in the case of quasi-stationary states it is convenient to express $a(t)$ in the following form

$$a(t) = a_{exp}(t) + a_{non}(t), \quad (50)$$

where $a_{exp}(t)$ is the exponential part of $a(t)$, that is $a_{exp} = N \exp[-it(E_\alpha^0 - \frac{i}{2}\gamma_\alpha^0)]$, (E_{WW}^0 is the energy of the system in the unstable state $|\alpha\rangle$ measured at the canonical decay times (when the exponential decay law is valid), N is the normalization constant), and $a_{non}(t)$ is the non-exponential part of $a(t)$. For times $t \sim \tau$, $|a_{exp}(t)| \gg |a_{non}(t)|$.

The transition time (or the crossover time) t_{as} can be found by solving the following equation,

$$|a_{exp}(t)|^2 = |a_{non}(t)|^2. \quad (51)$$

Long time properties of the survival probability $\mathcal{P}(t) = |a(t)|^2$ and the instantaneous energy $E_\alpha(t) = \Re[h(t)]$ of the system in the unstable state $|\alpha\rangle$ are relatively easy to find analytically for times $t \gg t_{as}$ even in the general case (see

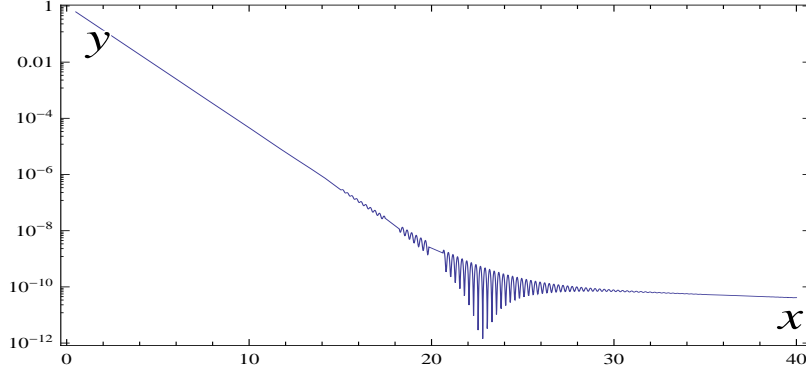


Fig. 1: Axes: $y = \mathcal{P}(t) = |a(t)|^2$ — the logarithmic scale, $x = t/\tau$. $\mathcal{P}(t)$ is the survival probability. The case $\frac{E_\alpha^0}{\gamma_\alpha^0} = 25$.

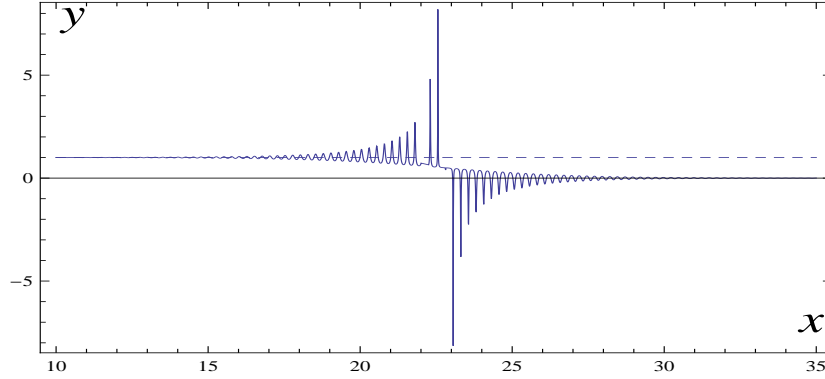


Fig. 2: Fluctuations of $E_\alpha(t) = \Re[h(t)]$ at the transition time region. Axes: $y = E_\alpha(t)/E_\alpha^0$, $x = t/\tau$. $E_\alpha^0 = \Re[h_{WW}]$, $\gamma_\alpha^0 = -2\Im[h_{WW}]$. The case $\frac{E_\alpha^0}{\gamma_\alpha^0} = 25$.

[20]). It is much more difficult to analyze these properties in the transition time region where $t \sim t_{as}$. Typical forms of $P(t)$ and $E_\alpha(t)$ are presented in Figs (1) and (2) respectively. Results presented in these figures were obtained numerically by means of the symbolic and numeric package "*Mathematica*": Using integral representation (49) of $a(t)$ the amplitude $a(t)$ was found numerically for a given $\omega(E)$, and then $|a(t)|^2$ and $\Re h(t)$ for $h(t)$ defined by (48). Calculations were performed for $\omega(E) = \frac{N}{2\pi} \Theta(E) \frac{\gamma_\alpha^0}{(E-E_\alpha^0)^2 + (\frac{\gamma_\alpha^0}{2})^2}$, $E_{min} = 0$.

Methods used in the asymptotic analysis allow one to find a form of $a(t) \equiv A_{\alpha\alpha}(t)$, (3), (49), for large t for all densities $\omega(E) = \omega_{\alpha\alpha}(E)$, corresponding

with the case ($Spec.(H) = [E_{min}, +\infty)$), for which the Fourier transform (3), (49) exists. For example, these calculations show that the amplitude $a_{non}(t)$ exhibits inverse power-law behavior at the late time region: $t \gg t_{as}$. The same can be done for the derivative of $a(t)$ and then using (48) a general asymptotic form of $h(t)$ can be found. It looks as follows

$$h(t)|_{t \rightarrow \infty} \simeq E_{min} + \left(-\frac{i}{t}\right) c_1 + \left(-\frac{i}{t}\right)^2 c_2 + \dots, \quad (52)$$

where $c_i = c_i^*$, $i = 1, 2, \dots$ (compare [20]). This last result means that

$$\Re[h(t)] \rightarrow E_{min} \text{ as } t \rightarrow \infty, \text{ and } \Im[h(t)] \rightarrow 0 \text{ as } t \rightarrow \infty. \quad (53)$$

4. Final remarks

The question arises: Can the effects described in Sec. 3, i.e., those presented in Figs (1) and (2) and those following from the relation (52) be observed? As it was mentioned earlier, the effect presented in Fig (1) was confirmed experimentally [11]. This means that effects presented in Fig (2) and following from (52) have to take place too.

In general, there is a chance to observe some of unstable particles, say ϕ , which survived at $t \sim t_{as}$ only if there is a source creating these particles in \mathcal{N}_ϕ^0 number such that

$$\mathcal{N}_\phi(t_{as}) = \mathcal{P}(t)|_{t \sim t_{as}} \mathcal{N}_\phi^0 \gg 1. \quad (54)$$

From (51) it follows that $\mathcal{P}(t_{as}) \simeq \exp[-\gamma_\phi^0 t_{as}]$. This means that if there is a source creating $\mathcal{N}_\phi^0 \gg \exp[+\gamma_\phi^0 t_{as}]$ unstable particles at the initial instant $t = t_0 = 0$, then a sufficiently large number $\mathcal{N}_\phi(t_{as})$ of unstable particles ϕ has to survive up to time t_{as} or later, and then the effect presented in Fig (2) should be observed. So in order to observe such effects one needs an unstable system having short t_{as} (as it was used in the experiment described in [11]) or one should find sources creating sufficiently large number \mathcal{N}_ϕ^0 unstable particles. Such sources are known from cosmology and astrophysics and the these effects should manifest itself there.

Effective Hamiltonians obtained in Sec. 2 are much more general and more accurate than LOY and WW approximations. Formulae for $V_{||}^{(1)}(t)$ and thus for $H_{||}(t) = PHP + V_{||}^{(1)}(t)$ derived in Sec. 2 seem to be a useful tool for a sufficiently accurate description of early time properties of complexes of unstable particles evolving in time. They work well for arbitrary $n < \infty$.

At times t of order the lifetime τ the time evolution of these complexes is well described by $V_{||} \stackrel{\text{def}}{=} \lim_{t \rightarrow \infty} V_{||}^{(1)}(t)$ obtained in Subsections 2.2 — 2.4.

Unfortunately these approximate formulae for $H_{\parallel}(t)$ are unable to describe correctly very late time properties of complexes of unstable particles, when $t \sim t_{as}$ or $t > t_{as}$. It can be done using the exact effective Hamiltonian of a form analogous to the one-dimensional effective Hamiltonian $h(t)$ given by (48) and using methods described in Sec. 3. Such a Hamiltonian acting in n -dimensional subspace has the following form

$$H_{\parallel}(t) = i \frac{\partial \mathbb{A}(t)}{\partial t} [\mathbb{A}(t)]^{-1}, \quad (55)$$

where $\mathbb{A}(t) = [A_{\alpha\beta}(t)]$ is $(n \times n)$ matrix, $\alpha, \beta = 1, 2, \dots, n$ and $A_{\alpha\beta}(t)$ are given by (3). Asymptotically late time properties of matrix elements $h_{\alpha\beta}(t)$ of this $H_{\parallel}(t)$ can be found using (3) and applying methods of asymptotic analysis to matrix elements $A_{\alpha\beta}(t)$ of $\mathbb{A}(t)$ and to $\frac{\partial}{\partial t} A_{\alpha\beta}(t)$.

Bibliography

1. V. F. Weisskopf and E.T. Wigner, Z. Phys., **63**, 54, (1930); **65**, 18, (1930).
2. T. D. Lee, R. Oehme and C. N. Yang, Phys. Rev., **106**, 340, (1957).
3. E. D. Comins and P. H. Bucksbaum, *Weak interactions of Leptons and Quarks*, (Cambridge University Press, 1983); T. P. Cheng and L. F. Li, *Gauge Theory of Elementary Particle Physics*, (Oxford, 1984).
4. M. Nowakowski, Int. J. Mod. Phys., **A 14**, 589, (1999).
5. L. Maiani, in *The Second DaΦne Physics Handbook*, vol. 1, Eds. L. Maiani, G. Pancheri and N. Paver, SIS — Pubblicazioni, INFN — LNF, Frascati, 1995; pp. 3 — 26.
6. K. Urbanowski and J. Piskorski, Found. Phys., **30**, 839, (2000).
7. S. Krylov, V. A. Fock, Zh. Eksp. Teor. Fiz. **17**, 93, (1947).
8. L. A. Khal'fin, Zh. Eksp. Teor. Fiz. **33**, 1371, (1957); [Sov. Phys. — JETP **6**, 1053, (1958)].
9. R. E. A. C. Paley, N. Wiener, *Fourier transforms in the complex domain*, American Mathematical Society, New York, 1934.
10. L. Fonda, G. C. Ghirardii and A. Rimini, Rep. on Prog. in Phys. **41**, 587, (1978).
11. C. Rothe, S. I. Hintschich and A. P. Monkman, Phys. Rev. Lett. **96**, 163601, (2006).
12. K. Urbanowski, Acta Phys. Polon. **B 14**, 485, (1983).
13. K. Urbanowski, Int. J. Mod. Phys. **A 7**, 6299, (1992); Phys. Lett. **A171**, 151, (1992).
14. K. Urbanowski, Phys. Rev. **A 50**, 2847, (1994).
15. K. Urbanowski, Int. J. Mod. Phys. **A 8**, 3721, (1993).
16. K. Urbanowski, Int. J. Mod. Phys. **A 10**, 1151, (1995).
17. W. Krolikowski and J. Rzewuski, Bull. Acad. Polon. Sci. **4**, 19, (1956); Nuovo. Cim. **B 25**, 739, (1975) and references therein.
18. K. Urbanowski, Int. J. Mod. Phys. **A 13**, 965, (1998).
19. K. Urbanowski, Cent. Eur. J. Phys. **7**, 696, (2009).
20. K. Urbanowski, Eur. Phys. J. **D 54**, 25, (2009).